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Exact solution of the Dirac–Eckart problem with spin and pseudospin symmetry*

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Abstract
Solving the s-wave Dirac equation for the Eckart potential with spin and pseudospin symmetry by using the supersymmetric quantum mechanics approach and function analysis method, we obtain the exact energy equation and corresponding two-component spinor wavefunctions. The restriction conditions of existing bound states are analysed.

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1. Introduction

It is well known that the exact solutions of the Dirac equation with mixed potentials play an important role in nuclear physics, such as in a realistic nuclear system where the nucleons are described to move independently in the relativistic mean field with the attractive scalar potential and repulsive vector potential. In recent years, there has been an increased interest in finding analytic solutions of the Dirac equation for some typical potentials under the special cases of spin symmetry and pseudospin symmetry [1–5]. Ginocchio [1] solved the Dirac equation for the triaxial, axial and spherical harmonic oscillators with spin symmetry. Lisboa *et al* [2, 3] investigated the analytical solutions of the Dirac equation for the generalized relativistic harmonic oscillator with spin symmetry and pseudospin symmetry. In [4], Chen *et al* solved the Dirac equation for two kinds of harmonic oscillator potentials with exact spin symmetry and pseudospin symmetry, respectively, and discussed the origin of pseudospin symmetry and its breaking in real nuclei in the relativistic mean field theory. Guo *et al* [5] studied the s-wave Dirac equation for the Woods–Saxon potential with spin and pseudospin symmetry. The condition of the difference between the vector and scalar potentials being a constant, i.e. $V(r) - S(r) =$ constant, leads to the spin symmetry that is relevant for

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mesons [6]. Based on the relativistic mean field theory, Ginocchio [7] showed that the pseudospin symmetry in nuclei could arise from nucleons moving in a relativistic mean field which has the near equality in magnitude of an attractive scalar potential $S(r)$, and repulsive vector potential $V(r)$, $S(r) \sim -V(r)$. Generally speaking, the pseudospin symmetry occurs for $V(r) + S(r) =$ constant in the Dirac equation [8]. Very recently, Alhaidari *et al* [9] have investigated in detail the physical interpretation on the three-dimensional Dirac equation in the cases of spin symmetry limitation ($V(r) - S(r) = 0$) and pseudospin symmetry limitation ($V(r)+S(r) = 0$). Ginocchio [10] showed that a Dirac Hamiltonian with the scalar and vector harmonic oscillator potentials in the case of $V(r) - S(r) = 0$ has not only a spin symmetry but also an U(3) symmetry, and a Dirac Hamiltonian with the scalar and vector harmonic oscillator potentials in the case of $V(r) + S(r) = 0$ has not only a pseudospin symmetry but also a pseudo-U(3) symmetry.

Recently, Zou *et al* [11] have investigated the Dirac equation with equally mixed potentials for the Eckart potential. The Eckart potential introduced by Eckart [12] has been widely used in physics [13] and chemical physics [14, 15]. The Eckart potential and its PT-symmetric version are the special cases of the five-parameter exponential-type potential model [16, 17]. In [11], the authors have only considered the case of the spin symmetry limitation, i.e., set the difference between the vector and scalar potentials to zero. Hence, it is of considerable interest to study the Dirac–Eckart problem with general spin symmetry and pseudospin symmetry.

In the present work, we investigate the analytic solutions of the Dirac equation for the Eckart potential with spin and pseudospin symmetry in terms of the supersymmetric quantum mechanics approach and function analysis method. The corresponding results for the case of the Dirac equation with equal scalar and vector Eckart potentials are only the special cases of the results given in this work.

2. Bound state solutions

The Dirac equation with both the scalar potential $S(r)$ and the vector potential $V(r)$ can be written as ($\hbar = c = 1$)

$$\{\boldsymbol{\alpha}\cdot \mathbf{p} + \beta[M + S(r)]\}\Psi(r) = [E - V(r)]\Psi(r), \tag{1}$$

where E denotes the energy, and M denotes the mass. For a particle in a spherical field, the total angular momentum operator $\mathbf{J}$, and spin–orbit matrix operator $K = -\beta(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$ commute with the Dirac Hamiltonian, where β, $\boldsymbol{\sigma}$ and $\mathbf{L}$ are, respectively, the Dirac matrix, Pauli matrix and orbital angular momentum. The eigenvalues of K are $k = \pm(j + 1/2)$ with – for aligned spin ($s_{1/2}$, $p_{3/2}$, etc) and + for unaligned spin ($p_{1/2}$, $d_{3/2}$, etc). The complete set of the conservative quantities can be taken as (H, K, J^2, J_z); the spinor wavefunctions can be classified according to their angular momentum j, k, and the radial quantum number n, and can be written in the form

$$\Psi_{nk} = \frac{1}{r}\begin{bmatrix} F_{nk}(r)Y^{l}_{jm}(\theta,\phi) \\ \mathrm{i}G_{nk}(r)Y^{\tilde{l}}_{jm}(\theta,\phi)\end{bmatrix}, \tag{2}$$

where the upper and lower radial functions $F_{nk}(r)$ and $G_{nk}(r)$ are real square-integral functions, $Y^{l}_{jm}(\theta,\phi)$ and $Y^{\tilde{l}}_{jm}(\theta,\phi)$ are the spherical harmonic functions, and m is the projection of angular momentum on the third axis. The orbital angular momentum quantum numbers l and $\tilde{l}$ refer to the upper and lower components, respectively. For a given $k = \pm1, \pm2, \ldots$, $j = |k| - 1/2$, $l = |k + 1/2| - 1/2$, $\tilde{l} = |k - 1/2| - 1/2$. Substituting equation (2) into equation (1), we obtain two coupled differential equations for the upper and lower radial wavefunctions $F_{nk}(r)$ and $G_{nk}(r)$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k}{r}\right)F_{nk}(r)=[M+E_{nk}+S(r)-V(r)]G_{nk}(r), \tag{3a}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}\right)G_{nk}(r)=[M-E_{nk}+S(r)+V(r)]F_{nk}(r). \tag{3b}$$

Eliminating $G_{nk}(r)$ in equation (3a) and $F_{nk}(r)$ in equation (3b), we obtain the following two second-order differential equations for the upper and lower components,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{l(l+1)}{r^2}-(M+E_{nk}-\Delta)(M-E_{nk}+\Sigma)+\frac{\frac{\mathrm{d}\Delta}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k}{r}\right)}{M+E_{nk}-\Delta}\right)F_{nk}(r)=0, \tag{4a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\tilde{l}(\tilde{l}+1)}{r^2}-(M+E_{nk}-\Delta)(M-E_{nk}+\Sigma)-\frac{\frac{\mathrm{d}\Sigma}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}\right)}{M-E_{nk}+\Sigma}\right)G_{nk}(r)=0, \tag{4b}$$

where $\Delta(r)=V(r)-S(r)$ and $\Sigma(r)=V(r)+S(r)$. In reducing equations (4a) and (4b), we have applied the relations $k(k+1)=l(l+1)$ and $k(k-1)=\tilde{l}(\tilde{l}+1)$, respectively.

(a) We consider the case of exact spin symmetry, i.e., $\frac{\mathrm{d}\Delta}{\mathrm{d}r}=0$ or $\Delta=C=$ constant, and reduce equation (4a) into the following form:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{l(l+1)}{r^2}-(M+E_{nk}-\Delta)(M-E_{nk}+\Sigma)\right)F_{nk}(r)=0. \tag{5}$$

From equation (5), we can see that the energy eigenvalues, E_{nk}, depend only on n and l, i.e., $E_{nk}=E(n,l(l+1))$. For $l\neq 0$, the states with $j=l\pm 1/2$ are degenerate. This is a SU(2) spin symmetry. We take the Eckart potential [12] as the $\Sigma(r)$,

$$\Sigma(r)=V_1\operatorname{cosech}^2\alpha r-V_2\coth\alpha r, \tag{6}$$

where α is a real positive parameter. With $\Delta(r)=V(r)-S(r)=C$, we obtain $V(r)=\frac{1}{2}(\Sigma(r)+C)$ and $S(r)=\frac{1}{2}(\Sigma(r)-C)$. Using the function $\Sigma(r)$ given in equation (6), we find that the potentials $V(r)$ and $S(r)$ have non-vanishing values at infinity. However, when $C=V_2$ and $C=-V_2$, the potentials $V(r)$ and $S(r)$ vanish at infinity, respectively. If $C=V_2=0$, then both potentials $V(r)$ and $S(r)$ vanish at infinity. Substituting equation (6) into equation (5), we obtain a Schrödinger-like equation for the s-wave ($l=0$, i.e., $k=-1$),

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\tilde{V}_1\operatorname{cosech}^2\alpha r+\tilde{V}_2\coth\alpha r\right)F_{n,-1}(r)=\tilde{E}F_{n,-1}(r), \tag{7}$$

where we have introduced the parameters $\tilde{V}_1=(M+E_{n,-1}-C)V_1$, $\tilde{V}_2=(M+E_{n,-1}-C)V_2$ and $\tilde{E}=E_{n,-1}^2-M^2+C(M-E_{n,-1})$. We use the supersymmetric quantum mechanics method and shape invariance approach to solve equation (7). The ground-state function for the upper radial component $F_{nk}(r)$ can be written in the form

$$F_{0,-1}(r)=\exp\left(-\int W(r)\,\mathrm{d}r\right), \tag{8}$$

where $W(r)$ is called a superpotential in supersymmetric quantum mechanics [18]. Substituting equation (8) into equation (7), we obtain the following equation for $W(r)$,

$$W^2(r)-\frac{\mathrm{d}W(r)}{\mathrm{d}r}=\tilde{V}_1\operatorname{cosech}^2\alpha r-\tilde{V}_2\coth\alpha r-\tilde{E}_0, \tag{9}$$

where $\tilde{E}_0$ is the ground-state energy. Equation (9) is a nonlinear Riccati equation. Putting the superpotential $W(r)$ as

$$W(r) = Q_1 + \frac{Q_2}{e^{2\alpha r} - 1}, \tag{10}$$

and substituting this expression into equation (9), we obtain the following relations:

$$Q_1^2 = -\tilde{E}_0 + \tilde{V}_2, \tag{11a}$$

$$2Q_1Q_2 + 2\alpha Q_2 = 4\tilde{V}_1 + 2\tilde{V}_2, \tag{11b}$$

$$Q_2^2 + 2\alpha Q_2 = 4\tilde{V}_1. \tag{11c}$$

Substituting equation (10) into equations (8), we obtain the ground-state function $F_{0,-1}(r)$

$$F_{0,-1}(r) = e^{-Q_1 r}\left(\frac{e^{2\alpha r}}{e^{2\alpha r} - 1}\right)^{\frac{Q_2}{2\alpha}}. \tag{12}$$

In this work, we deal with the bound state solutions, i.e., the radial part of the wavefunction Ψ_{nk} must satisfy the boundary conditions that $F_{nk}(r)/r$ becomes zero when $r \to \infty$, and $F_{nk}(r)/r$ is finite at $r = 0$. Obviously, only when $r \to \infty$, $F_{n,-1}(r)$ is finite, and $F_{n,-1}(r) = 0$ at the origin point $r = 0$; the radial wavefunction $F_{n,-1}(r)/r$ can satisfy the boundary conditions. In order to make the upper component $F_{0,-1}(r)$ satisfy the regularity conditions, we can obtain from equation (12) that $(Q_1 - Q_2) > 0$ and $Q_2 < 0$. Considering the boundary conditions and solving equations (11*a*)–(11*c*), we obtain

$$Q_1 = \frac{\tilde{V}_2}{Q_2} + \frac{Q_2}{2}, \tag{13a}$$

$$Q_2 = -\alpha\left[1 + \sqrt{1 + \frac{4\tilde{V}_1}{\alpha^2}}\right] = -2\alpha\eta, \tag{13b}$$

where the parameter η is defined as the combination of parameters, $\eta = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4\tilde{V}_1}{\alpha^2}}\right)$. With the help of equations (11*a*) and (13*a*), the corresponding ground-state energy and superpotential can be expressed as

$$\tilde{E}_0 = -\left[\frac{\tilde{V}_2}{Q_2} + \frac{Q_2}{2}\right]^2 + \tilde{V}_2, \tag{14}$$

$$W(r) = -\left(\frac{\tilde{V}_2}{Q_2} + \frac{Q_2}{2}\right) + \frac{Q_2}{e^{2\alpha r} - 1}. \tag{15}$$

Using the superpotential $W(r)$ given in equation (15), we can produce the following two supersymmetric partner potentials:

$$\tilde{V}_+(r) = W^2(r) + \frac{dW(r)}{dr} = \left(\frac{\tilde{V}_2}{Q_2} + \frac{Q_2}{2}\right)^2 + \frac{2\tilde{V}_2 + Q_2^2 - 2\alpha Q_2}{e^{2\alpha r} - 1} + \frac{Q_2^2 - 2\alpha Q_2}{(e^{2\alpha r} - 1)^2}, \tag{16a}$$

$$\tilde{V}_-(r) = W^2(r) - \frac{dW(r)}{dr} = \left(\frac{\tilde{V}_2}{Q_2} + \frac{Q_2}{2}\right)^2 + \frac{2\tilde{V}_2 + Q_2^2 + 2\alpha Q_2}{e^{2\alpha r} - 1} + \frac{Q_2^2 + 2\alpha Q_2}{(e^{2\alpha r} - 1)^2}. \tag{16b}$$

With the help of expressions (16a) and (16b), we can easily verify that the partner potentials $\tilde{V}_+(r)$ and $\tilde{V}_-(r)$ satisfy the following relationship,

$$\tilde{V}_+(r, a_0) = \tilde{V}_-(r, a_1) + R(a_1), \tag{17}$$

where $a_0 = Q_2$, a_1 is a function of a_0, i.e., $a_1 = f(a_0) = Q_2 - 2\alpha$, and the reminder $R(a_1)$ is independent of r, $R(a_1) = \left(\frac{\tilde{V}_2}{a_0} + \frac{a_0}{2}\right)^2 - \left(\frac{\tilde{V}_2}{a_1} + \frac{a_1}{2}\right)^2$. From equation (17), we can see that the two partner potentials $\tilde{V}_+(r)$ and $\tilde{V}_-(r)$ have the similar shapes and they are shape-invariant potentials in the sense of [19]. Applying the shape invariance approach [19], we can determine the energy spectra of the potential $\tilde{V}_-(r)$,

$$\tilde{E}_0^{(-)} = 0, \tag{18a}$$

$$\begin{aligned}\tilde{E}_n^{(-)} &= \sum_{k=1}^{n} R(a_k) = R(a_1) + R(a_2) + \cdots + R(a_n)\\ &= \left(\frac{\tilde{V}_2}{a_0} + \frac{a_0}{2}\right)^2 - \left(\frac{\tilde{V}_2}{a_1} + \frac{a_1}{2}\right)^2 + \left(\frac{\tilde{V}_2}{a_1} + \frac{a_1}{2}\right)^2 - \left(\frac{\tilde{V}_2}{a_2} + \frac{a_2}{2}\right)^2 + \cdots\\ &\quad + \left(\frac{\tilde{V}_2}{a_{n-1}} + \frac{a_{n-1}}{2}\right)^2 - \left(\frac{\tilde{V}_2}{a_n} + \frac{a_n}{2}\right)^2\\ &= \left(\frac{\tilde{V}_2}{a_0} + \frac{a_0}{2}\right)^2 - \left(\frac{\tilde{V}_2}{a_n} + \frac{a_n}{2}\right)^2\\ &= \left(\frac{\tilde{V}_2}{Q_2} + \frac{Q_2}{2}\right)^2 - \left(\frac{\tilde{V}_2}{Q_2 - 2n\alpha} + \frac{Q_2 - 2n\alpha}{2}\right)^2,\end{aligned} \tag{18b}$$

where the quantum number $n = 0, 1, 2, \ldots$. From equations (9) and (16b), we get the following relation:

$$\tilde{V}(r) = \tilde{V}_1 \,\mathrm{cosech}^2\, \alpha r + \tilde{V}_2 \coth \alpha r = \tilde{V}_-(r, a_0) + \tilde{E}_0. \tag{19}$$

With the help of equations (14), (18) and (19), we find the solution for $\tilde{E}$ in equation (7),

$$\tilde{E} = \tilde{E}_0 + \tilde{E}_n^{(-)} = -\left[\frac{\tilde{V}_2}{Q_2 - 2n\alpha}\right]^2 - \frac{(Q_2 - 2n\alpha)^2}{4}. \tag{20}$$

Substituting equation (13b) into equation (20) and using $\tilde{E} = E_{n,-1}^2 - M^2 + C(M - E_{n,-1})$, we obtain the energy equation for Eckart potential with spin symmetry in the Dirac theory,

$$M^2 - E_{n,-1}^2 - C(M - E_{n,-1}) = \alpha^2(n+\eta)^2 + \frac{(M + E_{n,-1} - C)^2 V_2^2}{4\alpha^2} \frac{1}{(n+\eta)^2}, \tag{21}$$

where the parameter η is given by $\eta = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4(M+E_{n,-1}-C)V_1}{\alpha^2}}\right)$.
If we take $C = 0$ and put $V_1 \to 2V_1$ and $V_2 \to 2V_2$, i.e., $S(r) = V(r) = \Sigma(r)$, the energy equation (21) becomes

$$M^2 - E_{n,-1}^2 = \alpha^2(n+\eta)^2 + \frac{(M + E_{n,-1})^2 V_2^2}{\alpha^2} \frac{1}{(n+\eta)^2}, \tag{22}$$

where $\eta = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8(M+E_{n,-1})V_1}{\alpha^2}}\right)$. Equation (22) is just expression (21) of [11], which is the energy equation for the Eckart potential with equally mixed potentials in the Dirac theory.
With the help of the superpotential $W(r)$ given in equation (15) and the ground-state function $F_{0,-1}(r)$ given in equation (12), we can yield the unnormalized excited state

spinor wavefunctions by using the operator method proposed by Dabrowska *et al* [20]. The corresponding normalization coefficients can be determined by applying the explicit recursion relations on the normalization coefficients of wavefunctions given in [21]. Recently, Fakhri and Chenaghlou [22] have also constructed the recursion relations on the coefficients of associated hypergeometric functions (not the wavefunctions).

Here, in order to obtain the unnormalized excited wavefunctions, we use the standard function analysis methods to solve equation (6). With the help of the energy spectrum expression (21), we rewrite equation (6) in the form of

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \tilde{V}_1 \,\mathrm{cosech}^2\, \alpha r - \tilde{V}_2 \coth \alpha r\right] F_{n,-1}(r) = \left[\alpha^2 (n+\eta)^2 + \frac{(M+E_{n,-1}-C)^2 V_2^2}{4\alpha^2 (n+\eta)^2}\right] F_{n,-1}(r). \tag{23}$$

Introducing a new variable $x = \mathrm{e}^{-2\alpha r}$ and writing the upper component $F_{n,-1}(r)$ as $f_{n,-1}(x)$, we can transform the equation (23) into the following form:

$$\left[x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\tilde{V}_1 x}{\alpha^2(1-x)^2} - \frac{\tilde{V}_2(1+x)}{4\alpha^2(1-x)} - \frac{(n+\eta)^2}{4} - \frac{(M+E_{n,-1}-C)^2 V_2^2}{16\alpha^4(n+\eta)^2}\right] f_{n,-1}(x) = 0. \tag{24}$$

The boundary conditions of $F_{n,-1}(r)$ are finite when $r \to \infty$ and $F_{n,-1}(r) = 0$ $(r = 0)$ can be turned to the boundary conditions of $f_{n,-1}(0) = 0$ $(r \to \infty)$ and $f_{n,-1}(1) = 0$ $(r = 0)$. Writing the function $f_{n,-1}(x)$ as $f_{n,-1}(x) = x^{\mu}(1-x)^{\eta} F(x)$, we find equation (24) can be reduced to the following hypergeometric differential equation

$$x(1-x)\frac{\mathrm{d}^2 F(x)}{\mathrm{d}x^2} + [2\mu + 1 - (2\mu + 2\eta + 1)x]\frac{\mathrm{d}F(x)}{\mathrm{d}x} - (2\eta + n)(2\mu - n)F(x) = 0, \tag{25}$$

where $\mu = \frac{1}{2}\left[n + \eta + \frac{(M+E_{n,-1}-C)V_2}{2\alpha^2}\frac{1}{n+\eta}\right]$. In reducing equation (25), we have applied $\tilde{V}_1 = (M + E_{n,-1} - C)V_1$ and $\tilde{V}_2 = (M + E_{n,-1} - C)V_2$. Equation (25) is the well-known differential equation satisfied by the hypergeometric function $F(2\eta + n, 2\mu - n, 2\mu + 1; x)$. Hence we obtain the upper component $F_{n,-1}(r)$ of the radial wavefunction corresponding to energy level $E_{n,-1}$,

$$F_{n,-1}(r) = \mathrm{e}^{-2\alpha\mu r}(1 - \mathrm{e}^{-2\alpha r})^{\eta} F(2\eta + n, 2\mu - n, 2\mu + 1; \mathrm{e}^{-2\alpha r}). \tag{26}$$

Substituting $F_{n,-1}(r)$ given in equation (26) into equation (3*a*) and considering that $k = -1$, we obtain the lower spinor component $G_{n,-1}(r)$ corresponding to the upper component $F_{n,-1}(r)$ and energy level $E_{n,-1}$,

$$\begin{aligned} G_{n,-1}(r) = {} & -\frac{1}{M + E_{n,-1} - C} \\ & \times \left[\left(\frac{1}{r}(1 - \mathrm{e}^{-2\alpha r})^{\eta} + 2\alpha\mu(1 - \mathrm{e}^{-2\alpha r})^{\eta} - 2\alpha\eta\, \mathrm{e}^{-2\alpha r}(1 - \mathrm{e}^{-2\alpha r})^{\eta - 1}\right)\right. \\ & \times \mathrm{e}^{-2\alpha\mu r} F(2\eta + n, 2\mu - n, 2\mu + 1; \mathrm{e}^{-2\alpha r}) + \frac{2\alpha(2\eta + n)(2\mu - n)}{2\mu + 1} \\ & \left.\times\, \mathrm{e}^{-2\alpha(\mu+1)r}\, (1 - \mathrm{e}^{-2\alpha r})^{\eta} F(2\eta + n + 1, 2\mu - n + 1, 2\mu + 2; \mathrm{e}^{-2\alpha r})\right]. \end{aligned} \tag{27}$$

Substituting $F_{n,-1}(r)$ and $G_{n,-1}(r)$ into equation (2), we can obtain the spinor wavefunction for the Eckart potential with spin symmetry in the Dirac theory.

In order to make the upper component $F_{n,-1}(r)$ and lower component $G_{n,-1}(r)$ satisfy the boundary conditions for the bound states, we observe equations (26) and (27) and obtain the restriction inequality conditions: $\eta > 1$ and $\mu > 0$. The energy level $E_{n,-1}$ is defined implicitly by the energy equation (21) which is a rather complicated transcendental equation having many solutions for a given value of n. For these solutions, we choose a suitable one, which can make the upper spinor component $F_{n,-1}(r)$ satisfy the restriction condition for the bound states. For example, we take $\alpha = 1$, $V_1 = 1$, $V_2 = 1$ and $M = 4$, when $n = 1$, and solve equation (21) by using the computer software MAPLE to yield the following values of $E_{1,-1}$: -3.0486, 1.2790. We choose $E_{1,-1} = 1.2790$ as the solution of equation (21), and find that the value of η is 2.85 and the value of μ is 2.27.

(b) For the case of exact pseudospin symmetry, i.e., $\frac{\mathrm{d}\Sigma}{\mathrm{d}r} = 0$ or $\Sigma = C = \text{constant}$, equation (4*b*) can be reduced to the following form:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\tilde{l}(\tilde{l}+1)}{r^2} + (M - E_{nk} + C)\Lambda(r) + \left(E_{nk}^2 - M^2 - C(M + E_{nk})\right)\right) G_{nk}(r) = 0. \tag{28}$$

From equation (28), we can see that the energy eigenvalues, E_{nk}, depend only on n and $\tilde{l}$, i.e., $E_{nk} = E(n, \tilde{l}(\tilde{l}+1))$. For $\tilde{l} \neq 0$, the states with $j = l \pm 1/2$ are degenerate. This is a SU(2) pseudospin symmetry. We take the Eckart potential [12] as the $\Delta(r)$,

$$\Delta(r) = V_1 \operatorname{cosech}^2 \alpha r - V_2 \coth \alpha r. \tag{29}$$

Substituting equation (29) into equation (28), we obtain a Schrödinger-like equation for the lower spinor component in the case of the s-wave ($\tilde{l} = 0$, i.e., $k = 1$),

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \tilde{V}_1 \operatorname{cosech}^2 \alpha r + \tilde{V}_2 \coth \alpha r\right) G_{n,1}(r) = \tilde{E} G_{n,1}(r), \tag{30}$$

where we have introduced the parameters $\tilde{V}_1 = -(M - E_{n,1} - C)V_1$, $\tilde{V}_2 = (M - E_{n,1} + C)V_2$ and $\tilde{E} = E_{n,1}^2 - M^2 - C(M + E_{n,1})$. Using the same procedure of solving equation (7), we obtain the energy equation for Eckart potential with pseudospin symmetry in the Dirac theory,

$$M^2 - E_{n,1}^2 + C(M + E_{n,1}) = \alpha^2 (n+\eta)^2 + \frac{(M - E_{n,1} + C)^2 V_2^2}{4\alpha^2} \frac{1}{(n+\eta)^2}, \tag{31}$$

where the quantum number $n = 0, 1, 2, 3, \ldots$. The unnormalized lower radial wavefunction is given by

$$G_{n,1}(r) = \mathrm{e}^{-2\alpha\mu r}(1 - \mathrm{e}^{-2\alpha r})^{\eta} F(2\eta + n, 2\mu - n, 2\mu + 1; \mathrm{e}^{-2\alpha r}), \tag{32}$$

where μ and η are given by, respectively,

$$\mu = \frac{1}{2}\left[n + \eta - \frac{(M - E_{n,1} + C)V_2}{2\alpha^2} \frac{1}{n+\eta}\right] \quad \text{and} \quad \eta = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4(M - E_{n,1} + C)V_1}{\alpha^2}}\right).$$

Substituting $G_{n,1}(r)$ given in equation (32) into equation (3*b*) and considering that $k = 1$, we obtain the upper spinor component $F_{n,1}(r)$ corresponding to the lower component $G_{n,1}(r)$ and energy level $E_{n,1}$,

$$F_{n,1}(r) = \frac{1}{M - E_{n,-1} + C} \times \left[\left(\frac{1}{r}(1 - \mathrm{e}^{-2\alpha r})^{\eta} - 2\alpha\mu(1 - \mathrm{e}^{-2\alpha r})^{\eta} + 2\alpha\eta\, \mathrm{e}^{-2\alpha r}(1 - \mathrm{e}^{-2\alpha r})^{\eta - 1}\right)\right.$$

$$\times\ \mathrm{e}^{-2\alpha\mu r} F(2\eta + n, 2\mu - n, 2\mu + 1; \mathrm{e}^{-2\alpha r}) - \frac{2\alpha(2\eta + n)(2\mu - n)}{2\mu + 1} \mathrm{e}^{-2\alpha(\mu+1)r}$$
$$\times (1 - \mathrm{e}^{-2\alpha r})^{\eta} F(2\eta + n + 1, 2\mu - n + 1, 2\mu + 2; \mathrm{e}^{-2\alpha r})\Big]. \tag{33}$$

For a given value of n, equation (31) has many solutions for the energy level $E_{n,1}$; however, we only choose a suitable one that can make the lower spinor component $G_{n,1}(r)$ and the upper spinor component $F_{n,1}(r)$ satisfy the restriction conditions for the bound states, i.e., $\eta > 1$ and $\mu > 0$. From equation (33), we can see that in the limit of pseudospin symmetry there are only bound negative energy states; otherwise the upper spinor component $F_{n,1}(r)$ will diverge. With the help of equation (27), we find that in the limit of spin symmetry there are no bound states with negative energy.

3. Conclusions

In conclusion, we may conclude that the Dirac equation for the Eckart potential with spin and pseudospin symmetry can be solved exactly for s-wave bound states with the help of the supersymmetric quantum mechanics approach and function analysis method. The energy equations and corresponding spinor wavefunctions for the s-wave bound states have been obtained analytically. Under the spin symmetry limitation, we recover the energy equation in the Dirac theory with equal scalar and vector Eckart potentials.

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